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1. Introduction

Let X denote a Banach space, let $\mathcal{L}(X)$ denote the set of all operators (bounded linear transformations) on X, and let $\mathcal{K}(X)$ denote the set of all compact operators on X (recall that an operator is said to be compact if the image of the unit ball has a compact closure). The essential norm $\|T\|_e$ of an operator T is the distance to the compact operators:

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(X)\} .$$

In 1965 Gohberg and Krein showed that this infimum is attained when X is a Hilbert space (see [16], Chap. II, Section 7, especially the proof of Cor. 7.1). However, they did not state this result explicitly and it was rediscovered in 1971 by Holmes and Kripke [21]. Independently, in 1972 Alfsen and Effros [4] introduced the notion of M-ideal in a Banach space, and they proved that if a subspace is an M-ideal, then for each element in the Banach space there exists a closest element in the subspace (see [4, Cor. 5.6]). In 1973 Hennefeld [18] showed that if X is any of the spaces c_0, l^p $(1 < p < \infty)$, then $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$; he did not, however, use this terminology and did not point out the connection with approximation theory. This was done in 1975 by Holmes, Scranton, and Ward [22]. They also showed that when dealing with M-ideals one does not have uniqueness of the element of best approximation.

Recently Mach and Ward [29] have given a constructive proof that each operator on l^p $(1 \leqq p < \infty)$ has a closest compact operator. The case $p = 1$ was handled by Lau [28] by another method; in addition he showed that $\mathcal{K}(l^1)$ is not an M-ideal in $\mathcal{L}(l^1)$. For the case $p = \infty$, Smith and Ward [37] showed that $\mathcal{K}(l^\infty)$ is not an M ideal in $\mathcal{L}(l^\infty)$. For further results see [3], [10], [14], [15], [23], [27], [39], and [40].

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In Section 2 we present a technique for producing closest compact operators of a certain form. For example, we show that every operator on l^p $(1 < p < \infty)$ whose matrix has all positive entries has a closest compact operator whose matrix also has only positive entries. A key ingredient is an inequality, and it seems to be of interest to identify those Banach spaces for which this Basic Inequality is valid.

In Section 3 we discuss the function space $H^\infty + C$ and apply the results of Section 2 to answer the question of whether each function f in L^∞ has a closest element in $H^\infty + C$. In addition, we show that this closest element is never unique for $f \notin H^\infty + C$; using this we show that the unit ball of the quotient space $L^\infty/(H^\infty + C)$ has no extreme points.

2. The existence of a closest compact operator

In this section we present a method for obtaining closest compact operators. In the next section these results are used to study the function space $H^\infty + C$.

Recall that a sequence $\{A_n\} \subset \mathfrak{L}(X)$ is said to converge to 0 in the strong operator topology (written $A_n \to 0$ (SOT)) if $\|A_n x\| \to 0$ for each vector x in X.

Definition. A Banach space X is said to satisfy the Basic Inequality if for each $T \in \mathfrak{L}(X)$ and each sequence $\{A_n\} \subset \mathfrak{L}(X)$ such that $A_n \to 0$ (SOT) and $A_n^* \to 0$ (SOT) the following is true: for each $\varepsilon > 0$, there exists N such that

$$\|T + A_N\| \leqq \varepsilon + \max\left(\|T\|, \|T\|_e + \|A_N\|\right).$$

Remark. If X satisfies the Basic Inequality and if T, $\{A_n\}$ and ε are as above, then there exists N such that

$$\|T + \beta A_N\| \leqq \varepsilon + \max\left(\|T\|, \|T\|_e + \beta\|A_N\|\right)$$

for all $\beta \in [0, 1]$. Indeed, if this were false then for each n there would exist $\beta_n \in [0, 1]$ such that

$$\|T + \beta_n A_n\| > \varepsilon + \max\left(\|T\|, \|T\|_e + \beta_n\|A_n\|\right). \tag{1}$$

But $\beta_n A_n \to 0$ (SOT) and $(\beta_n A_n)^* \to 0$ (SOT) and hence (1) contradicts the Basic Inequality.

The following theorem is the main tool of this paper.

THEOREM 1. *Let X be a Banach space that satisfies the Basic Inequality and let $T \in \mathfrak{L}(X) \sim \mathcal{K}(X)$. Let $\{T_n\} \subset \mathcal{K}(X)$ be a sequence of compact operators such that $T_n \to T$ (SOT) and $T_n^* \to T^*$ (SOT). Then there exists a sequence $\{a_n\}$ of non-negative real numbers such that $\sum a_n = 1$ and*

$$\|T - K\| = \|T\|_e ,$$

where $K = \sum a_n T_n$.

Remark. By the Principle of Uniform Boundedness the set $\{\|T_n\|\}$ is bounded; therefore the series $\sum a_n T_n$ is norm-convergent and thus K is compact.

Proof. Let $A_n = T - T_n$. Thus $A_n \to 0$ (SOT) and $A_n^* \to 0$ (SOT) and $\sup \|A_n\| < \infty$.

Claim. There exists an increasing sequence of positive integers $n(k)$ and a sequence α_k of positive real numbers such that

$$\sum_1^\infty \alpha_k = 1 , \tag{2}$$

$$\|\sum_1^N \alpha_k A_{n(k)}\| = \|T\|_e - \varepsilon_N , \tag{3}$$

for all N, where

$$\varepsilon_N = \|T\|_e/3^N .$$

We construct the $n(k)$ and α_k by induction. For $k = 1$, let $n(1) = 1$ and let $\alpha_1 > 0$ be such that $\|\alpha_1 A_1\| = \|T\|_e - \varepsilon_1$. Suppose $\alpha_1, \cdots, \alpha_N$ and $n(1), \cdots, n(N)$ have been chosen such that (3) holds. Choose $n(N+1) > n(N)$ such that

$$\|\sum_1^N \alpha_k A_{n(k)} + \beta A_{n(N+1)}\| \leqq \varepsilon_{N+1} + \max\left(\|\sum_1^N \alpha_k A_{n(k)}\|, \|\sum_1^N \alpha_k A_{n(k)}\|_e + \beta \|A_{n(N+1)}\|\right) \tag{4}$$

for every $\beta \in [0, 1]$. This is possible by the remark following the definition of the Basic Inequality.

Consider

$$\|\sum_1^N \alpha_k A_{n(k)} + \alpha A_{n(N+1)}\| . \tag{5}$$

As $\alpha \to \infty$, (5) becomes larger than $\|T\|_e - \varepsilon_{N+1}$. For $\alpha = 0$, (5) equals $\|T\|_e - \varepsilon_N$ by the induction hypothesis (3), and so is less than $\|T\|_e - \varepsilon_{N+1}$. Therefore there exists $\alpha_{N+1} > 0$ such that

$$\|\sum_1^{N+1} \alpha_k A_{n(k)}\| = \|T\|_e - \varepsilon_{N+1} . \tag{6}$$

Now we show that $\sum_1^{N+1} \alpha_k < 1$. From (6),

$$\|T\|_e - \varepsilon_{N+1} = \|(\sum_1^{N+1} \alpha_k) T - \sum_1^{N+1} \alpha_k T_{n(k)}\| \geqq (\sum_1^{N+1} \alpha_k) \|T\|_e$$

and so $\sum_1^{N+1} \alpha_k < 1$.

Take $\beta = \alpha_{N+1}$ in (4); if the maximum on the right hand side of (4) were attained by the first term then we would have

$$\|\sum_1^{N+1} \alpha_k A_{n(k)}\| \leqq \varepsilon_{N+1} + \|\sum_1^N \alpha_k A_{n(k)}\| .$$

By (6) and (3), the above becomes

$$\|T\|_e - \varepsilon_{N+1} \leqq \varepsilon_{N+1} + \|T\|_e - \varepsilon_N .$$

Thus $\varepsilon_N \leqq 2\varepsilon_{N+1}$, contradicting the definition of ε_N. Thus with $\beta = \alpha_{N+1}$, (4) becomes

$$\begin{aligned}\|\textstyle\sum_1^{N+1}\alpha_k A_{n(k)}\| &\leqq \varepsilon_{N+1} + \|\textstyle\sum_1^N \alpha_k A_{n(k)}\|_e + \alpha_{N+1}\|A_{n(N+1)}\| \\ &= \varepsilon_{N+1} + (\textstyle\sum_1^N \alpha_k)\|T\|_e + \alpha_{N+1}\|A_{n(N+1)}\| .\end{aligned}$$

Using (3) and letting $N \to \infty$ give

$$\|T\|_e = \|\textstyle\sum_1^\infty \alpha_k A_{n(k)}\| \leqq (\textstyle\sum_1^\infty \alpha_k)\|T\|_e .$$

Since we have already shown that $\sum \alpha_k \leqq 1$, we see that $\sum_1^\infty \alpha_k = 1$. This completes the proof of the claim.

Define $\{a_n\}$ by: $a_{n(k)} = \alpha_k$ and $a_i = 0$ if i is not of the form $n(k)$. Let

$$K = \textstyle\sum a_i T_i = \sum \alpha_k T_{n(k)} = T - \sum \alpha_k A_{n(k)} .$$

Thus

$$\|T - K\| = \|\textstyle\sum \alpha_k A_{n(k)}\| = \|T\|_e ,$$

which completes the proof of the theorem.

The following corollary shows that the operator K in Theorem 1 is never unique.

COROLLARY. *Let X, T, and $\{T_n\}$ be as in the statement of Theorem* 1. *Then there exist two sequences $\{a_n\}$, $\{b_n\}$ of non-negative real numbers such that $\sum a_n = \sum b_n = 1$ and*

$$\|T - K\| = \|T - K_1\| = \|T\|_e ,$$

where $K = \sum a_n T_n$, $K_1 = \sum b_n T_n$, and $K \neq K_1$.

Proof. Let $\{a_n\}$ and K be as given in the conclusion of Theorem 1. Then $T_n - K \to T - K$ (SOT) and $(T_n - K)^* \to (T - K)^*$ (SOT). Let $\mathcal{O}$ be a convex neighborhood of $T - K$ in the strong operator topology whose closure does not contain 0. By deleting a finite number of terms we may assume that $T_n - K \in \mathcal{O}$ for all n. By Theorem 1 there exists a sequence $\{b_n\}$ of non-negative numbers such that $\sum b_n = 1$ and

$$\|(T - K) - K'\| = \|T - K\|_e = \|T\|_e ,$$

where $K' = \sum b_n(T_n - K) = (\sum b_n T_n) - K$. Thus $K_1 = K + K' = \sum b_n T_n$ is also a closest compact operator to T. Since K' is an infinite convex combination of $\{T_n - K\} \subset \mathcal{O}$, we see that $K' \neq 0$. Thus $K \neq K_1$. Q.E.D.

The special case $p = 2$ of the following theorem is used in the next section.

THEOREM 2. *Let $1 < p < \infty$. Then l^p satisfies the Basic Inequality.*

Proof. Let $T \in \mathfrak{L}(l^p)$ and let $\{A_n\} \subset \mathfrak{L}(l^p)$ be such that $A_n \to 0$ (SOT) and $A_n^* \to 0$ (SOT). Let $\varepsilon > 0$ and let $\alpha_n = \max(\|T\|, \|T\|_e + \|A_n\|)$. If the desired inequality did not hold, then there would exist a sequence of unit vectors $\{x_n\} \subset l^p$ such that

$$(7) \qquad \|(T + A_n)x_n\| > \varepsilon + \alpha_n .$$

By passing to a subsequence (without relabeling) we may assume that there is a vector $x \in l^p$ such that $x_n \to x$ weakly. Let $y_n = x_n - x$. Then $x_n = x + y_n$, and $y_n \to 0$ weakly (and hence $Ty_n \to 0$ weakly). Hence

$$(8) \qquad (T + A_n)x_n = Tx + Ty_n + A_n y_n + A_n x .$$

Let E_n denote the natural projection of l^p onto the span of the first n coordinate vectors; let $F_n = 1 - E_n$ (where 1 denotes the identity operator l^p). Clearly $F_n \to 0$ (SOT) and $F_n^* \to 0$ (SOT). It follows that $\|F_n K\| \to 0$ and $\|KF_n\| \to 0$ for each compact operator K (we omit the proof). Since

$$\|T - K\| \geqq \|F_n(T - K)\| \geqq \|F_n T\| - \|F_n K\| ,$$

it can be shown that $\|F_n T\| \to \|T\|_e$.

Let δ be a small positive number to be specified later, and fix M such that $\|F_M Tx\| < \delta$, $\|F_M x\| < \delta$, and $\|F_M T\| < \|T\|_e + \delta$. Since $Ty_n \to 0$ weakly, we have $\|E_M Ty_n\| < \delta$ for all sufficiently large n.

Since E_M is compact and $A_n^* \to 0$ (SOT) we have $\|E_M A_n y_n\| \leqq \|E_M A_n\| \|y_n\| < \delta$ for all sufficiently large n.

From (8) we have

$$(T + A_n)x_n = E_M(Tx) + F_M(Ty_n + A_n y_n) + A_n x + F_M(Tx) + E_M(Ty_n + A_n y_n) .$$

Since $A_n \to 0$ (SOT), it follows that $\|A_n x\| < \delta$ for n sufficiently large. Thus for large n

$$(9) \qquad \|(T + A_n)x_n\| \leqq \|E_M(Tx) + F_M(Ty_n + A_n y_n)\| + 4\delta .$$

Since $E_M(Tx)$ and $F_M(Ty_n + A_n y_n)$ are vectors in l^p supported on disjoint subsets of the positive integers, we have

$$(10) \qquad \begin{aligned} \|E_M(Tx) + F_M(Ty_n + A_n y_n)\|^p &= \|E_M(Tx)\|^p + \|F_M(Ty_n + A_n y_n)\|^p \\ &\leqq \|T\|^p \|x\|^p + (\|F_M T\| + \|A_n\|)^p \|y_n\|^p \\ &\leqq \alpha_n^p \|x\|^p + (\|T\|_e + \delta + \|A_n\|)^p \|y_n\|^p \\ &\leqq (\alpha_n + \delta)^p (\|x\|^p + \|y_n\|^p) . \end{aligned}$$

Since $y_n \to 0$ weakly we have $\|E_M y_n\| < \delta$ for n sufficiently large; therefore

$$(11) \qquad \begin{aligned} ||x||^p + ||y_n||^p &= ||E_M x||^p + ||F_M x||^p + ||E_M y_n||^p + ||F_M y_n||^p \\ &\leqq ||E_M x||^p + ||F_M y_n||^p + 2\delta^p \\ &= ||E_M x + F_M y_n||^p + 2\delta^p \\ &= ||x - F_M x + y_n - E_M y_n||^p + 2\delta^p \\ &\leqq (||x_n|| + ||F_M x|| + ||E_M y_n||)^p + 2\delta^p \\ &\leqq (1 + 2\delta)^p + 2\delta^p . \end{aligned}$$

Applying (11) to (10) shows that

$$(12) \qquad ||E_M(Tx) + F_M(Ty_n + A_n y_n)||^p \leqq (\alpha_n + \delta)^p[(1 + 2\delta)^p + 2\delta^p] .$$

Applying (12) to (9) gives

$$(13) \qquad ||(T + A_n)x_n|| \leqq (\alpha_n + \delta)[(1 + 2\delta)^p + 2\delta^p]^{1/p} + 4\delta$$

for all sufficiently large n. Now choose δ small enough so that the right hand side of (13) is less than $\alpha_n + \varepsilon$, contradicting (7). This completes the proof of the theorem.

For operators on Hilbert space, ideas similar to the Basic Inequality were used by Berg [8].

Theorems 1 and 2 can be used to show that each operator on l^p $(1 < p < \infty)$ has a closest compact approximant. As noted in the introduction this result is already known; however, as the following corollary shows, our method gives additional information.

COROLLARY. *Let* $1 < p < \infty$, *let* $T \in \mathfrak{L}(l^p)$, *and let* (t_{ij}) *be the matrix representing* T *with respect to the usual basis. Then there exist numbers* (c_{ij}) *such that* $0 \leqq c_{ij} \leqq 1$ *and* $(c_{ij}t_{ij})$ *is the matrix representing a closest compact approximant to* T.

Proof. Let E_n denote the natural projection of l^p onto the span of the first n basis vectors. Let $T_n = E_n T E_n$. Then T_n is compact, $T_n \to T$ (SOT), and $T_n^* \to T^*$ (SOT). By Theorems 1 and 2 there exists a closest compact operator to T which is an infinite convex combination of $T_1, T_2, \cdots$. To complete the proof note that the matrix entry for T_n in position i, j is either 0 or t_{ij}. Q.E.D.

Note that the proof of the above corollary does not require very many properties of l^p, once the Basic Inequality is known to be satisfied. We state a more general theorem along these lines; further generalizations are possible (see [6]). Let X be a Banach space with a Schauder basis $\{e_k\}$ and let $\{\varphi_k\} \subset X^*$ be the associated coordinate functionals. Thus for each $x \in X$ we have

$$x = \sum_1^\infty \varphi_k(x)e_k .$$

Let E_n denote the projection of X onto the span of the first n basis vectors:

$$E_n x = \sum_1^n \varphi_k(x) e_k \ . \tag{14}$$

The basis $\{e_k\}$ is called a shrinking basis if the functionals $\{\varphi_k\}$ form a Schauder basis for X^*.

COROLLARY. *Let X be a Banach space which satisfies the Basic Inequality and which has a shrinking basis. Let $T \in \mathfrak{L}(X)$. Then there exists a compact operator K on X such that $\|T - K\| = \|T\|_e$.*

Proof. Let E_n be defined by (14), and let $T_n = E_n T$. Then T_n is compact and $T_n \to T$ (SOT). Since $\{e_k\}$ is a shrinking basis, we also have $T_n^* \to T^*$ (SOT). The result now follows from Theorem 1. Q.E.D.

This raises the question of identifying those Banach spaces X which satisfy the Basic Inequality. As we have seen, l^p $(1 < p < \infty)$ satisfies it. Also, l^1 satisfies it trivially, since it can be shown that if $\{A_n\} \subset \mathfrak{L}(l^1)$ and if $A_n^* \to 0$ (SOT), then $\|A_n\| \to 0$ (the proof is based on the representation of the operators as matrices; we omit the details). Also, a direct calculation shows that c_0 satisfies the Basic Inequality.

On the other hand, in [6] we show that if $1 \leqq p < \infty$ and $p \neq 2$, then L^p does not satisfy the Basic Inequality. Finally, we note that there exists a separable Banach space X, isomorphic to Hilbert space, which does not satisfy the Basic Inequality. Indeed, Holmes and Kripke [21] have constructed an equivalent norm on separable Hilbert space such that no non-compact operator on the space has a closest compact approximant. Of course, this renormed Hilbert space has a shrinking basis and so, by the last corollary, this space does not satisfy the Basic Inequality.

3. The distance to $H^\infty + C$

Let L^p denote the usual Lebesgue space of functions on the unit circle. Let H^p denote the subspace of L^p consisting of those functions whose Fourier coefficients of negative index vanish. Let C denote the space of continuous (complex-valued) functions on the unit circle. The linear span of H^∞ and C is denoted by $H^\infty + C$. It is known that $H^\infty + C$ is a closed subalgebra of L^∞ (see [33, p. 191]).

Among the subalgebras of L^∞ the algebra $H^\infty + C$ plays a special role. For example, every closed subalgebra of L^∞ that properly contains H^∞ must also contain $H^\infty + C$ (see [20]). The space $H^\infty + C$ also plays an important role in the theory of Toeplitz operators, in the study of the space of functions of vanishing mean oscillation (VMO), and in the factorization of L^∞

functions; see for example [5], [7], [11], [13], [32], [34].

We wish to solve the following problem raised by D. Sarason ([35], Problem 13, p. 197) and Adamjan, Arov, and Krein [2]: Does each function in L^∞ have a best approximant from $H^\infty + C$? Best H^∞ approximants always exist, since H^∞ is a weak-star closed subspace of L^∞ (in contrast, $H^\infty + C$ is weak-star dense in L^∞). It is also known that every L^∞ function has a best approximant from C; this is proved by a direct construction (for generalizations see [25], [26], and [31]).

The solution of this problem requires the introduction of Hankel operators. Let Q denote the orthogonal projection of L^2 onto $(H^2)^\perp$. For each $f \in L^\infty$ we define the Hankel operator $H_f: H^2 \to (H^2)^\perp$ by $H_f h = Q(fh)$ for $h \in H^2$. With respect to the usual orthonormal bases $\{z^n\}_0^\infty$ and $\{z^n\}_{-\infty}^{-1}$ for H^2 and $(H^2)^\perp$ respectively, the matrix for H_f is constant on the cross diagonals:

$$H_f \sim \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots & \\ a_{-3} & a_{-4} & \cdots & & \\ a_{-4} & \cdots & & & \\ \cdots & & & & \end{pmatrix},$$

where a_n is the n^{th} Fourier coefficient of f. Observe that if $h \in H^\infty$, then $H_f = H_{f+h}$ since only the Fourier coefficients of negative index are involved. In fact a theorem of Nehari [30] states that $\|H_f\| = \operatorname{dist}(f, H^\infty)$.

It is also true that $\|H_f\|_e = \operatorname{dist}(f, H^\infty + C)$. This follows from Theorem 0.1 of [1]; for completeness we present a more elementary proof. Let S denote the operator of multiplication by z on H^2 (the unilateral shift operator). If K is any compact operator and n is a positive integer then

$$\|H_f - K\| \geqq \|(H_f - K)S^n\| \geqq \|H_f S^n\| - \|KS^n\| .$$

Since $(S^n)^* \to 0$ (SOT) we have $\|KS^n\| \to 0$. Also, $H_f S^n = H_{fz^n}$. Hence

$$\begin{aligned}\|H_f - K\| &\geqq \overline{\lim}\, \|H_{fz^n}\| = \overline{\lim}\, \operatorname{dist}(fz^n, H^\infty) = \overline{\lim}\, \operatorname{dist}(f, z^{-n} H^\infty) \\ &\geqq \operatorname{dist}(f, H^\infty + C) ,\end{aligned}$$

since $H^\infty + C$ is an algebra. Thus $\|H_f\|_e \geqq \operatorname{dist}(f, H^\infty + C)$. In the reverse direction, we recall that if $h \in H^\infty + C$ then H_h is compact (this follows from the fact that a continuous function can be approximated by polynomials). Hence if $h \in H^\infty + C$, then

$$\|f - h\|_\infty \geqq \|H_f - H_h\| \geqq \|H_f - H_h\|_e = \|H_f\|_e ,$$

which completes the proof. In particular this yields Hartman's result [17] that H_f is compact if and only if $f \in H^\infty + C$.

THEOREM 3. *Let $f \in L^\infty$. Then there exists a compact Hankel operator K such that $\|H_f - K\| = \|H_f\|_e$.*

Proof. Let $\sigma_n f$ denote the n^{th} Cesaro mean of the Fourier series of f. Let $T = H_f$ and $T_n = H_{\sigma_n f}$. We claim that $T_n \to T$ (SOT). Indeed, for $h \in H^2$ we have

$$\|(T_n - T)h\|_2^2 = \|Q(f - \sigma_n f)h\|_2^2 \leqq \|(f - \sigma_n f)h\|_2^2 = \int |f - \sigma_n f|^2 |h|^2 .$$

Because $\|\sigma_n f\|_\infty \leqq \|f\|_\infty$ and $\sigma_n f \to f$ pointwise almost everywhere ([38], Chap. III, Thm. 3.9, p. 90), the integral above tends to zero by the Lebesgue dominated convergence theorem. Since $H_f^* h = (1 - Q)(\bar{f} h)$, a similar argument shows that $T_n^* \to T^*$ (SOT).

Theorem 1 and Theorem 2 (with $p = 2$) now give a best compact approximant $K = \sum a_n T_n$ to H_f. Let $g = \sum a_n \sigma_n f$; then $K = H_g$. Q.E.D.

In [1], Theorems 0.1 and 0.2, Adamjan, Arov, and Krein show that for each n, the distance from H_f to the set of operators of rank at most n is attained by a Hankel operator of rank at most n. Theorem 3 is the analogous result with the set of operators of rank at most n replaced by the set of compact operators.

It is necessary in the proof of Theorem 3 to use the Cesaro means rather than the partial sums $s_n f$ of the Fourier series because $H_f - H_{s_n f}$ does not necessarily tend to zero in the strong operator topology. However, each Cesaro mean $\sigma_n f$ is a convex combination of the partial sums of the Fourier series, and thus the function g obtained is an infinite convex combination of the partial sums $s_n f$.

We now answer the question posed earlier in this section.

THEOREM 4. *Let $f \in L^\infty$. Then there exists $h \in H^\infty + C$ such that $\|f - h\|_\infty = \operatorname{dist}(f, H^\infty + C)$.*

Proof. By Theorem 3 there exists a function $g \in H^\infty + C$ (actually the proof gives $g \in C$) such that $\|H_{f-g}\| = \|H_f\|_e$. From the remarks preceding Theorem 3 we know that $\|H_{f-g}\| = \operatorname{dist}(f - g, H^\infty)$ and $\|H_f\|_e = \operatorname{dist}(f, H^\infty + C)$. Let $d \in H^\infty$ be such that $\|(f - g) - d\|_\infty = \operatorname{dist}(f - g, H^\infty)$. Thus if $h = g + d$, then $h \in H^\infty + C$ and $\|f - h\|_\infty = \operatorname{dist}(f, H^\infty + C)$.

Q.E.D.

It would be of interest to know whether Theorem 4 remains valid if $H^\infty + C$ is replaced by an arbitrary closed subalgebra of L^∞ that contains H^∞.

As noted earlier each $f \in L^\infty$ has a best H^∞ approximant. This best approximant is sometimes unique; this is true, for example, when $f \in C$ (and hence when $f \in H^\infty + C$, [9], [36]). In fact, there are also functions not in $H^\infty + C$ that have unique best H^∞ approximants. In contrast, the following corollary shows that best $H^\infty + C$ approximants are never unique which answers a question raised in [2].

COROLLARY. *Let $f \in L^\infty$, $f \notin H^\infty + C$. Then there exist two different functions $h, h_1 \in H^\infty + C$ such that*

$$\|f - h\|_\infty = \|f - h_1\|_\infty = \operatorname{dist}(f, H^\infty + C) .$$

Proof. By the corollary to Theorem 1 we see that in Theorem 3 we could have concluded that there exist two distinct compact Hankel operators K, K_1 such that

$$\|H_f - K\| = \|H_f - K_1\| = \|H_f\|_e .$$

If $K = H_g$ and $K_1 = H_{g_1}$, then $g - g_1 \notin H^\infty$ (since $H_g \neq H_{g_1}$); the proof of Theorem 4, using g and g_1, now gives two distinct best $H^\infty + C$ approximants to f. Q.E.D.

The following corollary shows that $L^\infty/(H^\infty + C)$ is not the dual of any Banach space (in contrast to L^∞/H^∞, which is the dual of the space of functions in H^1 which have mean value zero). We would like to thank Donald Sarason for suggesting the proof. It is modeled on the proof of Koosis which characterizes extreme points of the closed unit ball of L^∞/H^∞ ([24], Theorem 4.1).

COROLLARY. *The closed unit ball of $L^\infty/(H^\infty + C)$ has no extreme points.*

Proof. Let $f + (H^\infty + C)$ be an element of the unit sphere of $L^\infty/(H^\infty + C)$. By Theorem 4 we can assume that $\|f\|_\infty = 1$. By the last corollary, there exists a function $h \in H^\infty + C$, $h \neq 0$, such that $\|f + h\|_\infty = 1$.

Let $|z| = 1$. Then

$$\left|f(z) + \frac{1}{2}h(z)\right| \leqq \frac{1}{2}|f(z)| + \frac{1}{2}|f(z) + h(z)| \leqq 1 .$$

Thus if $|f(z) + (1/2)h(z)| = 1$, then $|f(z)| = |f(z) + h(z)| = 1$; since the first inequality above is an equality, a calculation shows that $h(z)$ equals zero.

Define a function $g \in L^\infty$ by

$$g(z) = 1 - \left|f(z) + \frac{1}{2}h(z)\right| .$$

The previous paragraph shows that $g \geqq 0$ and that g is not identically zero. Thus there is a constant $c > 0$ and a non-trivial subset E of the circle such that $c\chi_E \leqq g$. The space $H^\infty + C$ contains no non-trivial characteristic functions (this follows by combining Cor. 6.42 of [12] with the third corollary of [19], p. 188) and so $f \pm c\chi_E + (H^\infty + C) \neq f + (H^\infty + C)$. Furthermore

$$\|f \pm c\chi_E + (H^\infty + C)\| \leqq \left\| f \pm c\chi_E + \frac{1}{2} h \right\|_\infty$$
$$\leqq \sup\left(\left| f(z) + \frac{1}{2} h(z) \right| + g(z) \right) = 1 .$$

Since

$$f + (H^\infty + C) = \frac{1}{2}(f + c\chi_E + (H^\infty + C)) + \frac{1}{2}(f - c\chi_E + (H^\infty + C)) ,$$

we conclude that $f + (H^\infty + C)$ is not an extreme point of the closed unit ball of $L^\infty/(H^\infty + C)$.

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REFERENCES

[1] V.M. ADAMJAN, D.Z. AROV, and M.G. KREIN, Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem, Matem. Sbornik **86** (128) (1971), 34-75; English translation: Math. USSR Sbornik **15** (1971), 31-73.

[2] ———, Approximation of L^∞ functions by means of functions of class $H^\infty+C$, Investigations in linear operators and the theory of functions. 99 unsolved problems in linear and complex analysis, edited by N.K. Nikolskii, V.P. Havin, and S.V. Hruscev, Zapiski nauc. seminar. LOMI, Vol. **81** (1978), 190-192 (Russian).

[3] CHARLES A. AKEMANN, GERT K. PEDERSEN, and JUN TAMIJAMA, Multipliers of C^*-algebras, J. Funct. Anal. **13** (1973), 277-301.

[4] ERIC M. ALFSEN and EDWARD G. EFFROS, Structure in real Banach spaces, Ann. of Math. **96** (1972), 98-173.

[5] SHELDON AXLER, Factorization of L^∞ functions, Ann. of Math. **106** (1977), 567-572.

[6] SHELDON AXLER, NICHOLAS JEWELL, and ALLEN SHIELDS, The essential norm of an operator and its adjoint, preprint.

[7] SHELDON AXLER, S. Y. CHANG, and DONALD SARASON, Products of Toeplitz operators, Integral Equations and Operator Theory **1** (1978), 285-309.

[8] I. DAVID BERG, On norm approximation of functions of operators in the Calkin algebra, preprint.

[9] F.F. BONSALL, Dual extremal problems in the theory of functions, J. London Math. Soc. **31** (1956), 105-110.

[10] C.K. CHUI, D.A. LEGG, P.W. SMITH, and J.D. WARD, On a question of Olsen concerning compact perturbations of operators, Michigan Math. J. **24** (1977), 119-127.

[11] KENNETH R. DAVIDSON, On operators commuting with Toeplitz operators modulo the compact operators, J. Funct. Anal. **24** (1977), 291-302.

[12] RONALD G. DOUGLAS, *Banach Algebra Techniques in Operator Theory*, Academic Press (1972).

[13] ———, Local Toeplitz operators, Proc. London Math. Soc. **36** (1978), 243-272.

[14] H. FAKHOURY, Approximation par des opérateurs compacts à valeurs dans $C(X)$, preprint.

[15] ———, Sur les M-idéaux dans certains espaces d'opérateurs et l'approximation par des opérateurs compacts, preprint.

[16] I.C. GOHBERG and M.G. KREIN, Introduction to the theory of linear non-selfadjoint

operators, Izdat. "Nauka", Moscow (1965); English translation: A. M. S. Translations of Math. Monographs, **18** (1969).

[17] PHILIP HARTMAN, On completely continuous Hankel matrices, Proc. A. M. S. **9** (1958), 862–866.

[18] JULIEN HENNEFELD, A decomposition for $B(X)^*$ and unique Hahn-Banach extensions, Pac. J. Math. **46** (1973), 197–199.

[19] KENNETH HOFFMAN, *Banach Spaces of Analytic Functions*, Prentice-Hall (1962).

[20] KENNETH HOFFMAN and I. M. SINGER, Maximal subalgebras of $C(\Gamma)$, Amer. J. Math. **79** (1957), 295–305.

[21] RICHARD B. HOLMES and BERNARD R. KRIPKE, Best approximation by compact operators, Indiana Univ. Math. J. **21** (1971), 255–263.

[22] RICHARD HOLMES, BRUCE SCRANTON, and JOSEPH WARD, Approximation from the space of compact operators and other M-ideals, Duke Math. J. **42** (1975), 259–269.

[23] ———, Uniqueness of commuting compact approximations, Trans. A. M. S. **208** (1975), 330–340.

[24] PAUL KOOSIS, Weighted quadratic means of Hilbert transforms, Duke Math. J. **38** (1971), 609–634.

[25] B. R. KRIPKE and R. B. HOLMES, Approximation of bounded functions by continuous functions, Bull. A. M. S. **71** (1965), 896–897.

[26] KA-SING LAU, Approximation by continuous vector-valued functions, preprint.

[27] ———, M-ideals and approximation by compact operators, preprint.

[28] ———, On a sufficient condition for proximity, preprint.

[29] JAROSLAV MACH and JOSEPH D. WARD, Approximation by compact operators on certain Banach spaces, preprint.

[30] ZEEV NEHARI, On bounded bilinear forms, Ann. of Math. **65** (1957), 153–162.

[31] C. OLECH, Approximation of set-valued functions by continuous functions, Colloq. Math. **19** (1968), 285–293.

[32] DONALD SARASON, Functions of vanishing mean oscillation, Trans. A. M. S. **207** (1975), 391–405.

[33] ———, Generalized interpolation in H^∞, Trans. A. M. S. **127** (1967), 179–203.

[34] ———, On products of Toeplitz operators, Acta. Scien. Math. (Szeged) **35** (1973), 7–12.

[35] ———, in *Spaces of Analytic Functions*, Kristiansand, Norway (1975), Springer-Verlag, Lecture Notes in Mathematics **512** (1976), 197.

[36] HAROLD S. SHAPIRO, Extremal problems for polynomials and power series, Dissertation, Massachusetts Institute of Technology (1952).

[37] R. R. SMITH and J. E. WARD, M-ideal structure in Banach algebras, J. Funct. Anal. **27** (1978), 337–349.

[38] A. ZYGMUND, *Trigonometric Series*, Second edition, Cambridge University Press (1959).

[39] C. A. AKEMANN and G. K. PEDERSEN, Ideal perturbations of elements in C-* algebras, Kobenhavns Universitet Matematisk Institut Preprint Series #17 (1976).

[40] C. L. OLSEN, Norms of compact perturbations of operators, Pac. J. Math. **68** (1977), 209–228.

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