Toeplitz Operators

Sheldon Axler

Abstract. This article discusses Paul Halmos's crucial work on Toeplitz operators and the consequences of that work.

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A *Toeplitz matrix* is a matrix that is constant on each line parallel to the main diagonal. Thus a Toeplitz matrix looks like this:

$$\begin{matrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \ddots \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{matrix}$$

In this article, Toeplitz matrices have infinitely many rows and columns, indexed by the nonnegative integers, and the entries of the matrix are complex numbers. Thus a Toeplitz matrix is determined by a two-sided sequence $(a_n)_{n=-\infty}^{\infty}$ of complex numbers, with the entry in row j, column k (for $j, k \geq 0$) of the Toeplitz matrix equal to a_{j-k}.

We can think of the Toeplitz matrix above as acting on the usual Hilbert space ℓ^2 of square-summable sequences of complex numbers, equipped with its standard orthonormal basis. The question then arises of characterizing the two-sided sequences $(a_n)_{n=-\infty}^{\infty}$ of complex numbers such that the corresponding Toeplitz matrix is the matrix of a bounded operator on ℓ^2. The answer to this question points toward the fascinating connection between Toeplitz operators and complex function theory.

This paper is an extension and modification of the author's article Paul Halmos and Toeplitz Operators, which was published in *Paul Halmos: Celebrating 50 Years of Mathematics*, Springer, 1991, edited by John H. Ewing and F. W. Gehring.

Let D denote the open unit disk in the complex plane and let σ denote the usual arc length measure on the unit circle ∂D, normalized so that $\sigma(\partial D) = 1$. For $f \in L^1(\partial D, \sigma)$ and n an integer, the n^{th} Fourier coefficient of f, denoted $\hat{f}(n)$, is defined by

$$\hat{f}(n) = \int_{\partial D} f(z)\overline{z^n}\, d\sigma(z).$$

The characterization of the Toeplitz matrices that represent bounded operators on ℓ^2 is now given by the following result.

Theorem 1. *The Toeplitz matrix corresponding to a two-sided sequence $(a_n)_{n=-\infty}^{\infty}$ of complex numbers is the matrix of a bounded operator on ℓ^2 if and only if there exists a function $f \in L^\infty(\partial D, \sigma)$ such that $a_n = \hat{f}(n)$ for every integer n.*

The result above first seems to have appeared in print in the Appendix of a 1954 paper by Hartman and Wintner [16], although several decades earlier Otto Toeplitz had proved the result in the special case of symmetric Toeplitz matrices (meaning that $a_{-n} = \overline{a_n}$ for each integer n). One direction of the result above is an easy consequence of adopting the right viewpoint. Specifically, the Hardy space H^2 is defined to be the closed linear span in $L^2(\partial D, \sigma)$ of $\{z^n : n \geq 0\}$. For $f \in L^\infty(\partial D, \sigma)$, the Toeplitz operator with symbol f, denoted T_f, is the operator on H^2 defined by

$$T_f h = P(fh),$$

where P denotes the orthogonal projection of $L^2(\partial D, \sigma)$ onto H^2. Clearly T_f is a bounded operator on H^2. The matrix of T_f with respect to the orthonormal basis $(z^n)_{n=0}^{\infty}$ is the Toeplitz matrix corresponding to the two-sided sequence $(\hat{f}(n))_{n=-\infty}^{\infty}$, thus proving one direction of Theorem 1.

Products of Toeplitz Operators

Paul Halmos's first paper on Toeplitz operators was a joint effort with Arlen Brown published in 1964 [5]. The Brown/Halmos paper set the tone for much of the later work on Toeplitz operators. Some of the results in the paper now seem easy, perhaps because in 1967 Halmos incorporated them into the chapter on Toeplitz operators in his marvelous and unique *A Hilbert Space Problem Book* [11], from which several generations of operator theorists have learned the tools of the trade. Multiple papers have been published in the 1960s, 1970s, 1980s, 1990s, and the 2000s that extend and generalize results that first appeared in the Brown/Halmos paper. Although it is probably the most widely cited paper ever written on Toeplitz operators, Halmos records in his automathography ([12], pages 319–321) that this paper was rejected by the first journal to which it was submitted before being accepted by the prestigious Crelle's journal.

The Brown/Halmos paper emphasized the difficulties flowing from the observation that the linear map $f \mapsto T_f$ is not multiplicative. Specifically, $T_f T_g$ is rarely equal to T_{fg}. Brown and Halmos discovered precisely when $T_f T_g = T_{fg}$. To state their result, first we recall that the Hardy space H^∞ is defined to be the set of functions f in $L^\infty(\partial D, \sigma)$ such that $\hat{f}(n) = 0$ for every $n < 0$. Note that a function $f \in L^\infty(\partial D, \sigma)$ is in H^∞ if and only if the matrix of T_f is a lower-triangular matrix. Similarly, the matrix of T_f is an upper-triangular matrix if and only if $\bar{f} \in H^\infty$. The Brown/Halmos paper gives the following characterization of which Toeplitz operators multiply well.

Theorem 2. *Suppose $f, g \in L^\infty(\partial D, \sigma)$. Then $T_f T_g = T_{fg}$ if and only if either $\bar{f}$ or g is in H^∞.*

As a consequence of the result above, the Brown/Halmos paper shows that there are no zero divisors among the set of Toeplitz operators:

Theorem 3. *If $f, g \in L^\infty(\partial D, \sigma)$ and $T_f T_g = 0$, then either $f = 0$ or $g = 0$.*

The theorem above naturally leads to the following question:

Question 1. *Suppose $f_1, f_2, \dots, f_n \in L^\infty(\partial D, \sigma)$ and*

$$T_{f_1} T_{f_2} \dots T_{f_n} = 0.$$

Must some $f_j = 0$?

Halmos did not put the question above in print, but I heard him raise and popularize it at a number of conferences. The Brown/Halmos paper shows that the question above has answer "yes" if $n = 2$. Several people extended the result to $n = 3$, but after that progress was painfully slow. In 1996 Kun Yu Guo [9] proved that the question above has an affirmative answer if $n = 5$. In 2000 Caixing Gu [8] extended the positive result to the case where $n = 6$. Recently Alexandru Aleman and Dragan Vukotić [1] completely solved the problem, cleverly showing that the question above has an affirmative answer for all values of n.

The Spectrum of a Toeplitz Operator

Recall that the spectrum of a linear operator T is the set of complex numbers λ such that $T - \lambda I$ is not invertible; here I denotes the identity operator. The Brown/Halmos paper contains the following result, which was the starting point for later deep work about the spectrum of a Toeplitz operator.

Theorem 4. *The spectrum of a Toeplitz operator cannot consist of exactly two points.*

In the best Halmosian tradition, the Brown/Halmos paper suggests an open problem as a yes/no question:

Question 2. *Can the spectrum of a Toeplitz operator consist of exactly three points?*

A bit later, in [10] (which was written after the Brown/Halmos paper although published slightly earlier) Halmos asked the following bolder question.

Question 3. *Does every Toeplitz operator have a connected spectrum?*

This has always struck me as an audacious question, considering what was known at the time. The answer was known to be "yes" when the symbols are required to be either real valued or in H^∞, but these are extremely special and unrepresentative cases. For the general complex-valued function, even the possibility that the spectrum could consist of exactly three points had not been eliminated when Halmos posed the question above.

Nevertheless, Harold Widom [19] soon answered Halmos's question by proving the following theorem (the essential spectrum of an operator T is the set of complex numbers λ such that $T - \lambda I$ is not invertible modulo the compact operators).

Theorem 5. *Every Toeplitz operator has a connected spectrum and a connected essential spectrum.*

Ron Douglas [7] has written that Widom's proof of the theorem above is unsatisfactory because "the proof gives us no hint as to why the result is true", but no alternative proof has been found.

Subnormal Toeplitz Operators

Recall that a linear operator T is called normal if it commutes with its adjoint ($T^*T = TT^*$). The Brown/Halmos paper gave the following characterization of the normal Toeplitz operators.

Theorem 6. *Suppose $f \in L^\infty(\partial D, \sigma)$. Then T_f is normal if and only if there is a real-valued function $g \in L^\infty(\partial D, \sigma)$ and complex constants a, b such that $f = ag+b$.*

One direction of the theorem above is easy because if g is a real-valued function in $L^\infty(\partial D, \sigma)$ then T_g is self-adjoint, which implies that $aT_g + bI$ is normal for all complex constants a, b.

A Toeplitz operator is called analytic if its symbol is in H^∞. The reason for this terminology is that the Fourier series of a function $f \in L^1(\partial D, \sigma)$, which is the formal sum

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n,$$

is the Taylor series expansion

$$\sum_{n=0}^{\infty} \hat{f}(n) z^n$$

of an analytic function on the unit disk if $\hat{f}(n) = 0$ for all $n < 0$.

An operator S on a Hilbert space H is called subnormal if there is a Hilbert space K containing H and a normal operator T on K such that $T|_K = S$. For example, if $f \in H^\infty$, then the Toeplitz operator T_f is subnormal, as can be seen by considering the Hilbert space $L^2(\partial D, \sigma)$ and the normal operator of multiplication by f on $L^2(\partial D, \sigma)$. Thus every analytic Toeplitz operator is subnormal.

All normal Toeplitz operators and all analytic Toeplitz operators are subnormal. These two classes of Toeplitz operators were the only known examples of subnormal Toeplitz operators in 1970 when Paul Halmos gave a famous series of lectures [14] in which he posed ten open problems in operator theory. One of Halmos's ten questions asked if there were any other examples:

Question 4. *Is every subnormal Toeplitz operator either normal or analytic?*

In 1979 Halmos described [15] what had happened to the ten problems in the years since they had been posed. The problem about Toeplitz operators was still unsolved, but Halmos's question had stimulated good work on the problem. Several papers had been written with partial results providing strong evidence that the question above had an affirmative answer.

In the spring of 1983 I believed that the time was right for a breakthrough on this problem, so I organized a seminar at Michigan State University to focus on this problem. We went through every paper on this topic, including a first draft of a manuscript by Shun-Hua Sun. Sun claimed to have proved that no nonanalytic Toeplitz operator can lie in a certain important subclass of the subnormal operators. There was a uncorrectable error in the proof (and the result is false), but Sun had introduced clever new ideas to the subject. His proof worked for all but a single family of operators, and thus this particular family was an excellent candidate for a counter-example that no one expected to exist.

The Spring quarter ended and I sent a copy of my seminar notes to Carl Cowen at Purdue University. When I returned from a long trip abroad, I found a letter from Cowen, who had amazingly answered Halmos's question (negatively!) by proving that each operator in the suspicious family singled out by Sun's work is a subnormal Toeplitz operator that is neither normal nor analytic. Here is what Cowen had proved, where we are abusing notation and thinking of f, which starts out as a function on D, as also a function on ∂D (just extend f by continuity to ∂D):

Theorem 7. *Suppose $b \in (0, 1)$. Let f be a one-to-one analytic mapping of the unit disk onto the ellipse with vertices $\frac{1}{1+b}$, $\frac{-1}{1+b}$, $\frac{i}{1-b}$, and $\frac{-i}{1-b}$. Then the Toeplitz operator with symbol $f + b\bar{f}$ is subnormal but is neither normal nor analytic.*

I told my PhD student John Long about Cowen's wonderful result, although I did not show Long the proof. Within a week, Long came back to me with a beautiful and deep proof that was shorter and more natural than Cowen's. Because there was now no reason to publish Cowen's original proof, Cowen and Long decided to publish Long's proof in a joint paper [6]. Thus the contributions to that paper are as follows: Cowen first proved the result and provided the crucial knowledge of the correct answer, including the idea of using ellipses; the proof in the paper is due to Long. At no time did the two authors actually work together.

The Symbol Map

Paul Halmos's second major paper on Toeplitz operators was a joint effort with José Barría that was published in 1982 [4]. The main object of investigation is the Toeplitz algebra $\mathcal{T}$, which is defined to be the norm-closed algebra generated by all the Toeplitz operators on H^2. The most important tool in the study of $\mathcal{T}$ is what is called the symbol map φ, as described in the next theorem.

Theorem 8. *There exists a unique multiplicative linear map $\varphi : \mathcal{T} \to L^\infty(\partial D, \sigma)$ such that $\varphi(T_f) = f$ for every $f \in L^\infty(\partial D, \sigma)$.*

The surprising point here is the existence of a multiplicative map on $\mathcal{T}$ such that $\varphi(T_f) = f$ for every $f \in L^\infty(\partial D, \sigma)$. Thus

$$\varphi(T_f T_g) = \varphi(T_f)\varphi(T_g) = fg$$

for all $f, g \in L^\infty(\partial D, \sigma)$. The symbol map φ was discovered and exploited by Douglas ([7], Chapter 7).

The symbol map φ was a magical and mysterious homomorphism to me until I read the Barría/Halmos paper, where the authors actually construct φ (as opposed to Douglas's more abstract proof).

Here is how the Barría/Halmos paper constructs φ: The authors prove that if $S \in \mathcal{T}$, then S is an asymptotic Toeplitz operator in the sense that in the matrix of S, the limit along each line parallel to the main diagonal exists. Consider a Toeplitz matrix in which each line parallel to the main diagonal contains the limit of the corresponding line from the matrix of S. The nature of the construction ensures that this Toeplitz matrix represents a bounded operator and thus is the matrix of T_f for some $f \in L^\infty(\partial D, \sigma)$. Starting with $S \in \mathcal{T}$, we have now obtained a function $f \in L^\infty(\partial D, \sigma)$. Define $\varphi(S)$ to be f. Then φ is the symbol map whose existence is guaranteed by Theorem 8.

A more formal statement of the Barría/Halmos result is given below. Here we are using φ as in Theorem 8. Thus the point here is that we can actually construct the symbol map φ.

Theorem 9. *Suppose $S \in \mathcal{T}$ and the matrix of S with respect to the standard basis of H^2 is $(b_{j,k})_{j,k=0}^\infty$. Then for each integer n, the limit (as $j \to \infty$) of $b_{n+j,j}$ exists. Let*

$$a_n = \lim_{j \to \infty} b_{n+j,j}$$

and let

$$f = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where the infinite sum converges in the norm of $L^2(\partial D, \sigma)$. Then $f \in L^\infty(\partial D, \sigma)$ and $\varphi(S) = f$.

The Barría/Halmos construction of φ is completely different in spirit and technique from Douglas's existence proof. I knew Douglas's proof well—an idea that I got from reading it was a key ingredient in my first published paper [2]. But until the Barría/Halmos paper came along, I never guessed that φ could be explicitly constructed or that so much additional insight could be squeezed from a new approach.

Compact Semi-commutators

An operator of the form $T_f T_g - T_{fg}$ is called a semi-commutator. As discussed earlier, the Brown/Halmos paper gave a necessary and sufficient condition on functions $f, g \in L^\infty(\partial D, \sigma)$ for the semi-commutator $T_f T_g - T_{fg}$ to equal 0. One of the fruitful strands of generalization stemming from this result involves asking for $T_f T_g - T_{fg}$ to be small in some sense. In this context, the most useful way an operator can be small is to be compact.

In 1978 Sun-Yung Alice Chang, Don Sarason, and I published a paper [3] giving a sufficient condition on functions $f, g \in L^\infty(\partial D, \sigma)$ for $T_f T_g - T_{fg}$ to be compact. This condition included all previously known sufficient conditions. To describe this condition, for $g \in L^\infty(\partial D, \sigma)$ let $H^\infty[g]$ denote the smallest norm-closed subalgebra of $L^\infty(\partial D, \sigma)$ containing H^∞ and g. The Axler/Chang/Sarason paper showed that if $f, g \in L^\infty(\partial D, \sigma)$ and

$$H^\infty[\bar{f}] \cap H^\infty[g] \subset H^\infty + C(\partial D),$$

then $T_f T_g - T_{fg}$ is compact.

We could prove that the condition above was necessary as well as sufficient if we put some additional hypotheses on f and g. We conjectured that the condition above was necessary without the additional hypotheses, but we were unable to prove so.

A brilliant proof verifying the conjecture was published by Alexander Volberg [18] in 1982. Combining Volberg's result of the necessity with the previous result of the sufficiency gives the following theorem.

Theorem 10. *Suppose $f, g \in L^\infty(\partial D, \sigma)$. Then $T_f T_g - T_{fg}$ is compact if and only if $H^\infty[\bar{f}] \cap H^\infty[g] \subset H^\infty + C(\partial D)$.*

A key step in Volberg's proof of the necessity uses the following specific case of a theorem about interpolation of operators that had been proved 26 years earlier by Elias Stein ([17], Theorem 2).

Theorem 11. *Let $d\mu$ be a positive measure on a set X, and let v and w be positive measurable functions on X. Suppose S is a linear operator on both $L^2(v d\mu)$ and $L^2(w d\mu)$, with norms $\|S\|_v$ and $\|S\|_w$, respectively. If $\|S\|_{\sqrt{vw}}$ denotes the norm of S on $L^2(\sqrt{vw}\, d\mu)$, then*

$$\|S\|_{\sqrt{vw}} \leq \sqrt{\|S\|_v \, \|S\|_w}.$$

When I received a preprint of Volberg's paper in Spring 1981 I told Paul Halmos about the special interpolation result that it used. Within a few days

Halmos surprised me by producing a clean Hilbert space proof of the interpolation result above that Volberg had needed. Halmos's proof (for the special case of Theorem 11) was much nicer than Stein's original proof. With his typical efficiency, Halmos put his inspiration into publishable form quickly and submitted the paper to the journal to which Volberg has submitted his article. I ended up as the referee for both papers. It was an unusual pleasure to see how a tool used in one paper had led to an improved proof of the tool. Halmos's short and delightful paper [13] containing his proof of the interpolation result was published in the same issue of the *Journal of Operator Theory* as Volberg's paper.

Remembering Paul Halmos

I would like to close with a few words about my personal debt to Paul Halmos. Paul is my mathematical grandfather. His articles and books have been an important part of my mathematical heritage. I first met Paul for a few seconds when I was a graduate student, and then for a few minutes when I gave my first conference talk right after receiving my PhD. Four years later I got to know Paul well when I spent a year's leave at Indiana University. Later when Paul became Editor of the *American Mathematical Monthly*, he selected me as one of the Associate Editors. Still later Paul and I spent several years working together as members of the Editorial Board for the Springer series Graduate Texts in Mathematics, Undergraduate Texts in Mathematics, and Universitext.

Paul is one of the three people who showed me how to be a mathematician (the other two are my wonderful thesis advisor Don Sarason, who was Paul's student, and Allen Shields). Watching Paul, I saw how an expert proved a theorem, gave a talk, wrote a paper, composed a referee's report, edited a journal, and edited a book series. I'm extremely lucky to have had such an extraordinary model.

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Sheldon Axler
Mathematics Department
San Francisco State University
San Francisco, CA 94132, USA
e-mail: `axler@sfsu.edu`